# Localization transition in incommensurate non-Hermitian systems

Amin Jazaeri and Indubala I. Satija*
*Department of Physics and School of Computational Sciences, George Mason University, Fairfax, Virginia 22030*
(Received 14 September 2000; published 27 February 2001)

A class of one-dimensional lattice models with an incommensurate complex potential $V(\theta)=2[\lambda_r\cos(\theta)+i\lambda_i\sin(\theta)]$ is found to exhibit a localization transition at $|\lambda_r|+|\lambda_i|=1$. This transition from extended to localized states manifests itself in the behavior of the complex eigenspectum. In the extended phase, states with real eigenenergies have a finite measure, and this measure goes to zero in the localized phase. Furthermore, all extended states exhibit real spectra provided $|\lambda_r|\geqslant|\lambda_i|$. Another interesting feature of the system is the fact that the imaginary part of the spectrum is sensitive to the boundary conditions *only at the onset to localization*.

DOI: 10.1103/PhysRevE.63.036222 PACS number(s): 05.45.−a, 75.30.Kz, 64.60.Ak, 64.60.Fr

Incommensurate systems such as the Harper equation [1] provide an important class of models exhibiting both extended and localized states in one dimension. In this paper, we study the localization transition in systems with competing length scales where the underlying potential is complex. The systems under investigation are described by the class of lattice tight binding models (TBM's),

$$\psi_{n+1}+\psi_{n-1}+2[\lambda_r\cos(\theta_n)+i\lambda_i\sin(\theta_n)]\psi_n=E\psi_n, \tag{1}$$

where $\theta_n=2\pi\sigma n+\alpha$. Here $\alpha$ is a constant phase factor and $\sigma$ is an irrational number, which we choose to be the golden mean. This lattice model describes a system where the period of the potential is incommensurate with the periodicity of the lattice. For $\lambda_i=0$, this reduces to a Harper equation, which exhibits a localization transition at $|\lambda_r|=1$. Recently, non-Hermitian systems have been the subject of various theoretical [2–5] and experimental [6,7] studies. In certain one-dimensional random systems, where all states for the Hermitian problem are localized, the addition of a complex vector potential has been shown [3] to result in a delocalization of states, accompanied by the eigenvalues becoming complex. The system we investigate exhibits both extended and localized states, and hence facilitates a study of non-Hermiticity in both these phases. By investigating how the non-Hermiticity alters the localization transition as well as the eigenenergies in the complex plane, we are able to study the correlation between the nature of the eigenspectrum and the transport characteristics of the model. It is implicit in the literature [8] that non-Hermiticity corresponds to dissipation and decoherence, and as such we arguably explore the impact of these effects on the localization transition. Two interesting limits that we explore in detail are the case of purely imaginary potential $\lambda_r=0$, and the case where the real and imaginary parts are equal $(\lambda_r=\lambda_i)$, which can be described as the strong and weak dissipative limits, respectively. Further, we also study the case of non-Hermitian lattice models with real spectra. This is an interesting problem, since certain complex potentials in quantum mechanics are known to have real spectra provided the potentials exhibit parity and time reversal (PT) symmetry [2]. Here we seek the criterion for a real spectrum in non-Hermitian lattice models exhibiting localization transitions.

We study periodic boundary conditions (PBC's), antiperiodic boundary conditions (APBC's), and free boundary conditions to investigate the sensitivity to the boundary effects. As expected, only extended states are sensitive to the boundary conditions, and this sensitivity to the boundary effects can be used to distinguish extended and localized states. We use $\Delta E_r$, which is the real part of difference in the eigenenergies between PBC's and APBC's, summed over all states, to distinguish extended and localized states: The extended states are characterized by a finite value of $\Delta E_r$ while in the localized phase $\Delta E_r=0$, reflecting its insensitivity to changes at the boundaries. Our detailed numerical study, based on sensitivity to various boundary conditions and wave functions, shows that the non-Hermitian system exhibits a localization transition at $|\lambda_r|+|\lambda_i|=1$. This implies that the Hermitian and non-Hermitian parts of the potential carry the same weight in determining the transport characteristics of the model. Figure 1 shows $\Delta E_r$ and $\Delta E_i$ for the weak dissipation limit $\lambda_r=\lambda_i=\lambda$, which is discussed in detail below. A rather intriguing result is that the imaginary part of the $\Delta E$ is sensitive to the boundary effects only at the onset to the localization transition. This result, found to be true for other parameter values, implies that the lifetime of the metastable system depends upon the boundary conditions only at the transition point.

In order to explore the relationship between the localization character and the behavior of the eigenenergies, we have extensively studied the eigenspectrum in the two-parameter $(\lambda_r,\lambda_i)$ space. Figures 2–4 show the variation in eigenenergies with the parameters for some special cases. These studies suggest that it is the localized phase of the non-Hermitian lattice model [Eq. (1)] that is characterized by a complex spectrum, in contrast to previous results [3] where the spectra become complex when the localized states become delocalized. In the extended phase, the spectrum is real provided $|\lambda_r|\geqslant|\lambda_i$. Furthermore, for $|\lambda_r|<|\lambda_i|$, a number of extended states with real eigenenergies have finite measures. Therefore, the fraction of states with real spectra is finite in extended phase, and vanishes in localized phase. This provides a new order parameter for the localization transition, as shown in Fig. 5.

*Email address: isatija@gmu.edu

 

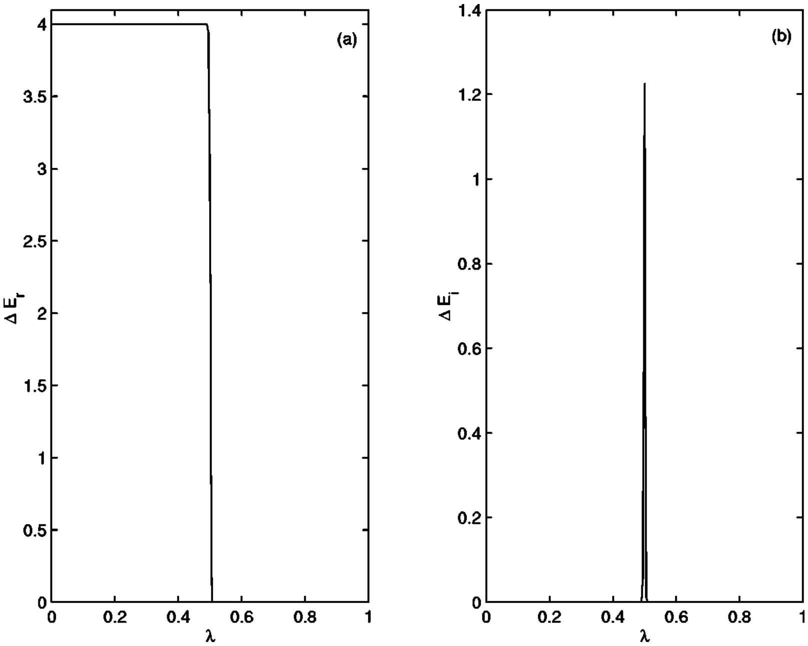


FIG. 1. (a) and (b) show $\Delta E_r$ and $\Delta E_i$ vs $\lambda \equiv \lambda_r = \lambda_i$ for $\sigma = 377$ and 610, respectively. $\Delta E_i$ appears to be related to the derivative of $\Delta E_r$.

In contrast with earlier results [3], where a non-Hermitian vector potential was found to delocalize the localized states of the random system, the addition of a non-Hermitian potential to a Harper model which exhibits both extended and localized states does not alter the localization character of the system. Furthermore, our results associate a complex spectrum with the localized states, in marked contrast with the earlier result where a complex spectrum implied delocalization. This is one of the central results of our analysis. It implies that previous results relating complex eigenvalues and delocalization must be understood as specific to the type of system investigated, namely, a random system with a non-Hermitian vector potential, and may not describe the generic property of non-Hermitian systems exhibiting localization.

We now discuss two limiting cases: the strong and weak dissipation limits. The first case corresponds to a purely imaginary potential $(\lambda_r = 0)$. It is interesting to note that a model with a purely imaginary potential exhibits a duality very similar to that of the purely Hermitian problem [9]. Under the Fourier transformation (FT)

$$\psi_n = \sum e^{i\theta_n m} \phi_m , \tag{2}$$

the TBM [Eq. (1)] with $\lambda \equiv \lambda_i$ reduces to

$$\phi_{m+1} + \phi_{m-1} + \frac{2i}{(\lambda)} \cos(\theta_m) \phi_m = -\frac{iE}{\lambda} \phi_m . \tag{3}$$

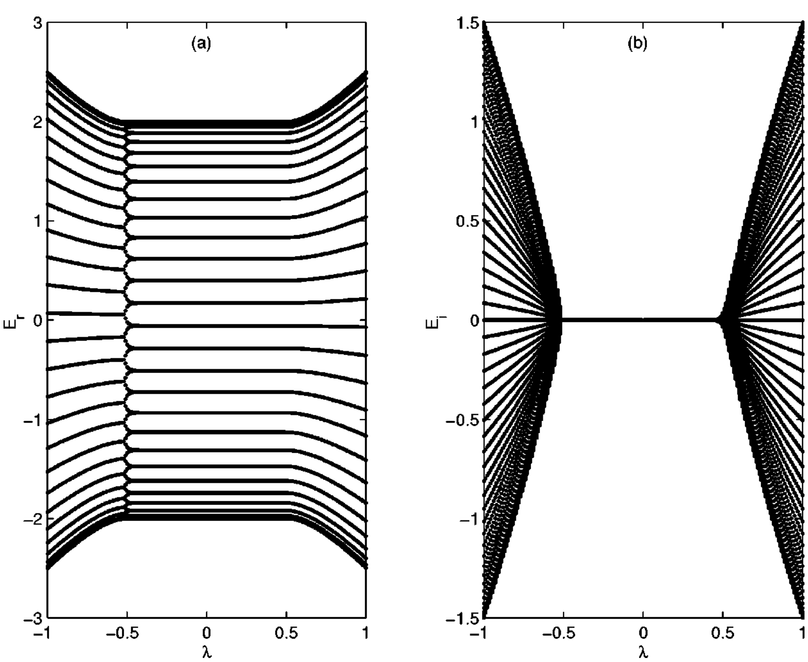


FIG. 2. (a) and (b) show the variation in eigenvalues as a function of $\lambda \equiv \lambda_r = \lambda_i$ for PBC for $\sigma = 34$ and 55, respectively. The extended phase of this non-Hermitian system exhibits a real spectrum. Note that in contrast to the Harper equation, there is a bending and merging of levels at the transition. Furthermore, unlike the Harper equation, the spectrum is not symmetric about $\lambda = 0$.

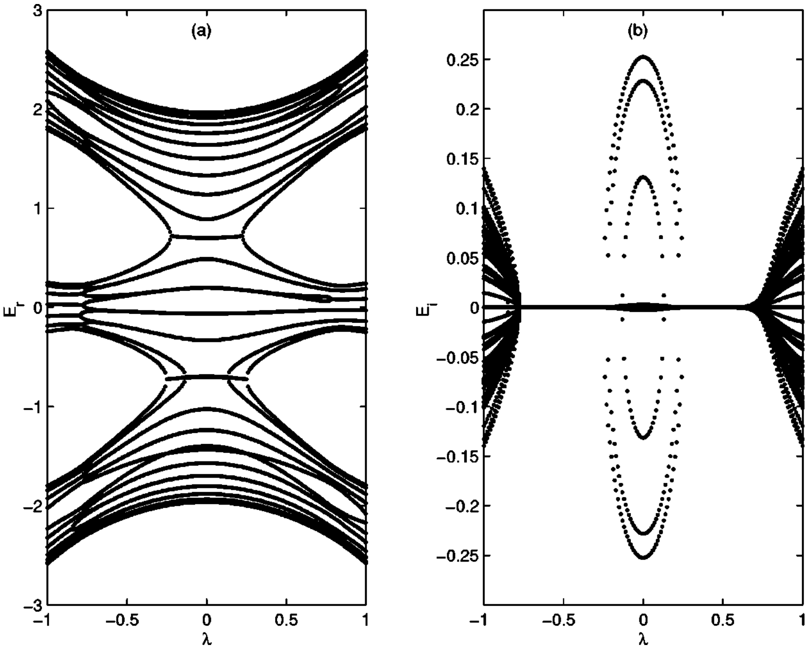


FIG. 3. (a) and (b) show the variation in eigenvalues as a function of $\lambda \equiv \lambda_r$ for fixed $\lambda_i = 0.25$ at PBC's for $\sigma = 34$ and 55, respectively. In the extended phase all states have real eigenenergies provided $\lambda_r \geqslant \lambda_i$.

Comparing this with the original model, we obtain

$$E^*(\lambda) = -\frac{i}{\lambda} E\left(\frac{1}{\lambda}\right). \tag{4}$$

This implies that the real and imaginary parts of the eigenenergies are related as $E_r(\lambda) = (1/\lambda) E_i(1/\lambda)$. Therefore, the case of the purely imaginary potential has the interesting property that the localization transition interchanges the real and imaginary parts of the spectrum. At the onset to localization ($\lambda = 1$) the model is self-dual, with $E_r = E_i$. Figure 6 shows eigenenergies at some values of the parameter in the extended phase. Due to duality these figures also show the spectrum in the localized phase with the interchange of $E_r$ and $E_i$. At the onset of localization, the spectrum with $E_r = E_i$ resembles that of the Harper equation. It is intriguing that even in this strong dissipative limit, extended states with real spectra have a finite measure. In this lattice model with competing length scales, it therefore appears that extended states are essential for obtaining real eigenenergies. This is to be compared with earlier formal results, where PT symmetry was a key for obtaining real spectrum for complex potentials.

We next discuss the weak dissipation limit, $\lambda_r = \lambda_i \equiv \lambda$, described by the following TBM:

$$\psi_{n+1} + \psi_{n-1} + 2\lambda e^{i\theta_n} \psi_n = E \psi_n. \tag{5}$$

As shown in Fig. 1, the system exhibits a localization transition at $\lambda = .5$. The localization threshold is half of that of the Harper equation, because both the real and imaginary

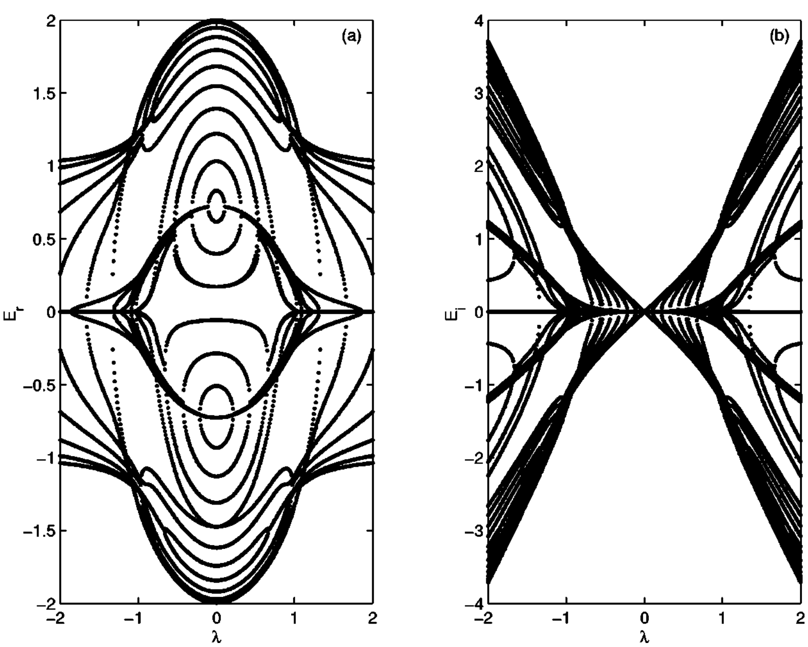


FIG. 4. Same as Figs. 2 and 3 for the purely imaginary potential $\lambda_r = 0$.

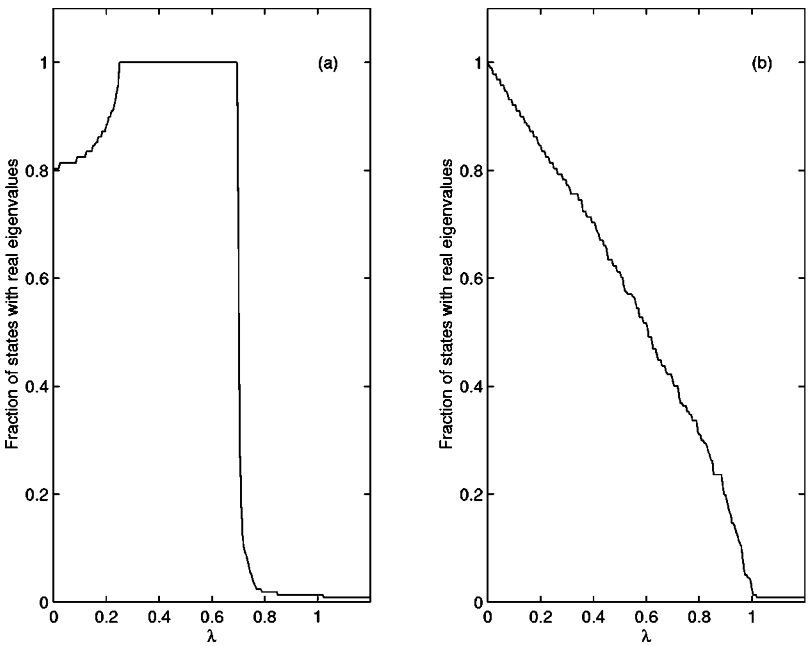


FIG. 5. Fraction of states with real eigenenergies as a function of λ for $\lambda_i = 0.25$ (a) and $\lambda_r = 0$ (b). Here $\sigma = 233$ and 377. The steps seen in this plot are due to finite size effects.

parts of the potential contributes toward localization. In the extended phase, the spectrum is found to be real and identical to that of the $\lambda = 0$ limit, namely, $E = 2\cos(\theta_n)$. This explains the constant value of $\Delta E_r$ in the extended phase, as seen in Fig. 1. In the localized phase the eigenenergies are complex, and appear to be described by the following expressions:

$$E_r = 2\cosh(\gamma)\cos(\theta_n), \tag{6}$$

$$E_i = 2\sinh(\gamma)\sin(\theta_n). \tag{7}$$

Here $\gamma$ is the inverse localization length of the system, which is found to be equal to the corresponding value for the Harper equation $\gamma = \log(2\lambda)$. In the limit $\lambda \to \infty$, the eigenenergies lie on a circle. Therefore, the localized phase is metastable, with the lifetime determined by the localization length.

The FT of model (5) can be analyzed further as transformation (2) reduces the tridiagonal matrix to the following triangular matrix:

$$2\cos(\theta_m)\phi_m + 2\lambda\phi_{m-1} = E\phi_m. \tag{8}$$

With PBC's, the eigenvalues of this triangular matrix are the solution of the following algebraic equation:

$$(2\lambda)^N = \prod [E - 2\cos(\theta_n)]. \tag{9}$$

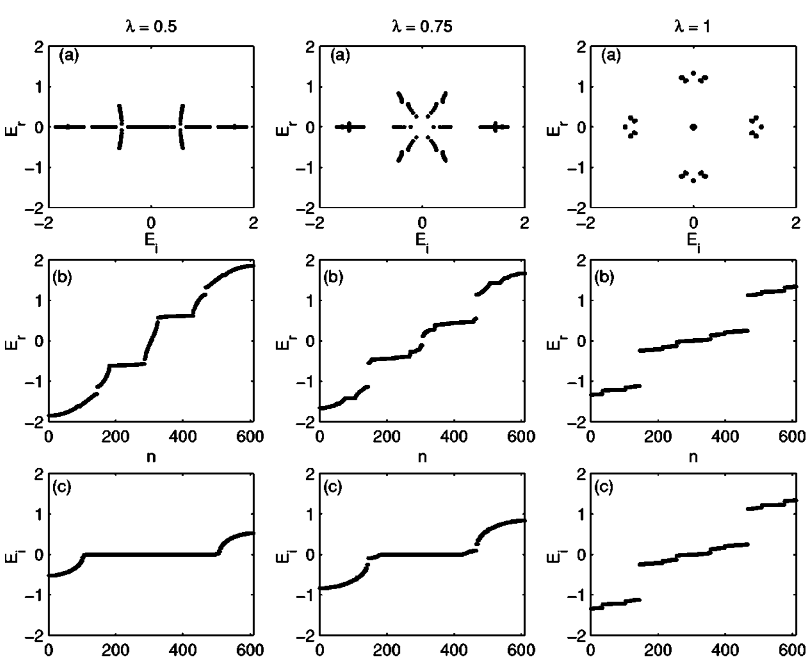


FIG. 6. Spectrum for different values of λ for $\sigma = 55$ and 89, respectively, for a purely imaginary potential. The three vertical columns correspond to $\lambda = 0.5$, 0.75, and 1, respectively. The three rows show (a) $E_r$ vs $E_i$ (b) $E_r$ vs $n$ and (c) $E_i$ vs $n$ (sorted independently of the real part).

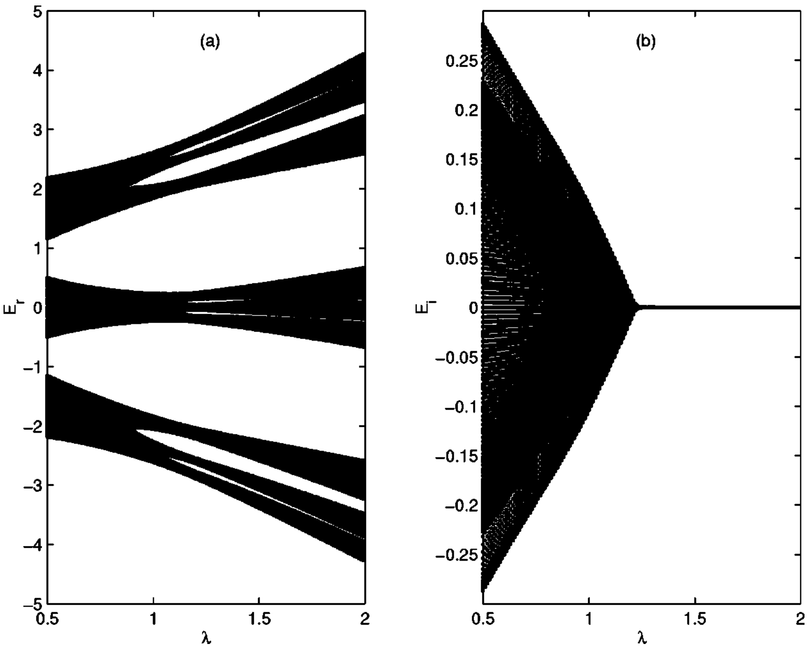


FIG. 7. (a) and (b) show the real and imaginary parts of the spectrum for the model described by Eq. (10) with $g=0.2$ and $\sigma=144$ and 233, respectively. In contrast to the case where non-Hermiticity appears in the diagonal part, the localized phase is characterized by real eigenvalues.

For $\lambda<0.5$, as $N\to\infty$, we obtain $E=2\cos(\theta_n)$, which is found to be identical with numerically obtained spectrum of model (5). For $\lambda>0.5$, the FT of the model allows real solutions: $E=2\cos(\theta_n)+2\lambda$. These real energies were not found to be the solutions of model (5). The FT of the model also exhibits complex solutions. It is easy to see that in the $\lambda\to\infty$ limit, the algebraic equation (9) has a solution where $E/(2\lambda)$ lies on the unit circle, which is also the solution for model (5). This indicates a deep relationship between the spectra of model (5) and its FT — the details have so far proven elusive.

Another aspect of model (5) is that at the onset to localization $\lambda=0.5$, the FT of the model describes the strong coupling limit of the fluctuations of the Harper equation once the exponentially decaying part is factored out [10]. This also describes the Ising model at the onset to long range order for $E=0$ [12]. This limit has been shown to be universal using renormalization methods [10] as well as more rigorous analytic tools [11]. This result therefore establishes the multifractal character of the FT of the wave function at the onset of the localization transition. It should be noted that the eigenspectrum remains continuous at the localization transition, in contrast to the Harper equation, which is characterized by a singular-continuous spectrum at the transition. Therefore, the $\lambda_r=\lambda_i$ limit of model (1) provides a class of incommensurate systems where the eigenspectrum is gapless and remains continuous except in the localized phase.

In contrast with earlier results on localization in non-Hermitian systems [3], the system studied here associates a complex spectrum with the localized phase. This difference appears to be related to the fact that the non-Hermiticity appears in the diagonal part of the potential while the non-Hermitian vector potential studied earlier [3] affects the off-diagonal part of the lattice model. To confirm this view, we studied a Harper equation with a non-Hermitian off-diagonal term:

$$e^{g}\psi_{n+1}+e^{-g}\psi_{n-1}+2\lambda\cos(\theta_n)\psi_n=E\psi_n. \qquad (10)$$

The parameter $g$ is related to the complex vector potential. As shown in Fig. 7, the localized phase of this model is associated with the real eigenspectrum, as was the case for the random potential studied in Ref. [4].

In summary, the localization transition of the Harper equation remains unaffected by the non-Hermitian perturbation. The correlation between the nature of eigenstates and the behavior of the energy spectrum in the complex plane depends upon whether the non-Hermiticity appears in the diagonal or off-diagonal part of the model. For a lattice model with a non-Hermitian diagonal potential, the case that has been studied in detail here, a weakly dissipative system is characterized by real eigenenergies in extended phase. As the strength of the non-Hermitian potential increases, the number of extended states with real eigenenergies decrease, approaching zero at the onset to localization. In the localized phase, states with complex spectra have a full measure. The localized phase is metastable, with state lifetimes determined by the localization length. The question of real eigenvalues is determined by both the transport character of the states as well as the amount of dissipation. An interesting result is that extended states with real eigenenergies always have a positive measure, while localized states with real energies have a zero measure. Therefore, the measure of real eigenenergies provides an order parameter for the localization-delocalization transition.

One future aspect of study is the extension of these results to a classical nonintegrable (perhaps a kicked) system, thus investigating the effect of dissipation on nonintegrable systems exhibiting localization. Furthermore, it should be noted that Eq. (1) is the fermion representation of isotropic $XY$ spin-$\frac{1}{2}$ chain in a complex magnetic field which is spatially modulating [12]. The consequences of a localization transition with a complex spectrum on the magnetic properties of the system is another interesting open question. Finally, a localization transition in incommensurate tight-binding lattice models corresponds to a transition to strange nonchatoic attractors (SNA's) [13] in quasiperiodically driven maps. Therefore, the results of this paper may have important implications in the study of SNA's in complex maps.

The research of I.I.S. was supported by a grant from National Science Foundation Grant No. DMR 097535. We would like to thank Arjendu Pattanayak for useful discussions.


[1] P. G. Harper, Proc. Phys. Soc., London, Sect. A **68**, 874 (1955). For a review, see J. B. Sokoloff, Phys. Rep. **126**, 189 (1985).

[2] C. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).

[3] N. Hatano and D. R. Nelson, Phys. Rev. Lett. **77**, 570 (1996); Phys. Rev. B **56**, 8651 (1997).

[4] P.G. Silvestrov, Phys. Rev. Lett. **82**, 3140 (1999); e-print, cond-mat/9804093.

[5] M. V. Berry, J. Phys. A **31**, 3493 (1998); M. Berry and O. Dell, **31**, 2093 (1998).

[6] K. Keller, M. K. Obertaler, S. Bernet, R. Abfalterer, J. Schmiedmayer, and A. Zeilinger, Phys. Rev. Lett. **79**, 3327 (1997); **77**, 4980 (1996).

[7] K. Schaadt and A. Kudrolli, Phys. Rev. E **60**, 7203 (1999).

[8] D. Ferry and J. R. Barker, Appl. Phys. Lett. **74**, 582 (1999).

[9] S. Aubry and G. André, Ann. Isr. Phys. Soc. **3**, 133 (1980).

[10] J. Ketoja and I. Satija, Phys. Rev. Lett. **75**, 2762 (1995).

[11] Ben Mestel, Andy Osbaldestin, and B. Winn (unpublished).

[12] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. (N.Y.) **16**, 407 (1961); I. I. Satija, Phys. Rev. B **48**, 3511 (1993); **49**, 3391 (1994).

[13] A. Bondeson, E. Ott, and T. M. Antonsen, Phys. Rev. Lett. **55**, 2103 (1985); J. Ketoja and I. Satija, Physica D **1761**, 1 (1997).